A host of traveling waves in a model of three-dimensional water-wave dynamics

Robert L. Pego[1] José Raúl Quintero[2]

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Abstract

We describe traveling waves in a basic model for three-dimensional water-wave dynamics in the weakly nonlinear long-wave regime. Small solutions that are periodic in the direction of translation (or orthogonal to it) form an infinite-dimensional family. We characterize these solutions through spatial dynamics, by reducing a linearly ill-posed mixed-type initial-value problem to a center manifold of infinite dimension and codimension. A unique global solution exists for arbitrary small initial data for the two-component bottom velocity, specified along a single line in the direction of translation (or orthogonal to it). A dispersive, nonlocal, nonlinear wave equation governs the spatial evolution of bottom velocity.

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1 Introduction

To describe steadily propagating nonlinear gravity waves on the free surface of a three-dimensional ideal fluid, it seems natural to seek shapes with simple symmetry, e.g., doubly periodic, or localized, say. Hammack *et al.* [7, 6] have expressed a belief that periodic water waves may tend to form hexagonal patterns. They have produced such waves experimentally, and have described such patterns using theta-function solutions of the Kadomtsev-Petviashvili equation, which models long waves of moderate amplitude propagating mainly in one direction with weak transverse variation. As Craig and Nicholls [3] have recently pointed out, however, in the exact water wave equations, the question of existence of doubly periodic gravity waves exhibits the problem of small divisors, and remains open.

In this paper we aim to develop an approach to describing steady water waves through spatial dynamics, relaxing the assumption of periodicity in two directions. We consider a basic isotropic model for waves in shallow water that shares some crucial features with the exact water wave equations. For this model we are able to describe all small waves that translate steadily with supercritical speed and that are periodic in the direction of translation (or orthogonal to it). There is in fact a host of such waves — they form an infinite-dimensional family. The family may be parametrized by the bottom-velocity profile along any single line in the direction of translation (or orthogonal to it). Fixing a supercritical wave speed (meaning, in dimensional terms, a speed greater than $\sqrt{gh}$, where g is the acceleration of

[1]Department of Mathematics & Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742 USA

[2]Departamento de Matemáticas, Universidad del Valle, A. A. 25360, Cali, Colombia.

gravity and h the undisturbed fluid depth), for an arbitrary bottom-velocity profile small in a suitable Sobolev space, there corresponds a unique globally defined traveling wave.

Simpler models of water waves exhibit the same phenomenon, as was discussed in [8]. The simplest is the linear wave equation for the surface elevation η, given in nondimensional form by $\eta_{tt} = \eta_{xx} + \eta_{yy}$. One can find many traveling-wave solutions $\eta = f(x - ct, y)$ translating with supercritical speed $|c| > 1$, by solving the wave equation $f_{yy} = (c^2 - 1)f_{\xi\xi}$ with given arbitrary initial data for the wave slope (η_x, η_y) along the single line $y = 0$ for example. The fact that $|c| > 1$ does not violate Huyghens' principle; these solutions are simply superpositions of two "scissoring" one-dimensional wave trains that propagate with unit speed in directions oblique to the x-axis.

Two other simple models considered in [8] were: (i) the exact linearized water wave equations for an inviscid irrotational fluid without surface tension; and (ii) the Kadomtsev-Petviashvili equation in the KP-II case. In each case, arbitrary small data for wave slope along a line determine a unique traveling wave with given supercritical speed. For the exact linearized equations, these solutions can be regarded as superpositions of a continuum of plane waves that propagate obliquely to the x-axis but translate along it with the same speed c. What is intriguing about the case of the KP equation is that one expects its nonlinearity to disrupt the delicate superposition principle that apparently permits such a large family of steadily traveling waves to exist. Since the KP equation is integrable, however, one could speculate that the persistence of this large family of waves is due to some nonlinear superposition principle special to integrable systems.

In this paper, we will demonstrate that the infinite-dimensional nature of the family of steadily propagating water wave patterns is really a robust phenomenon, for a class of nonlinear model equations such as were derived by Benney and Luke [2] to describe the oblique interaction of water waves at high angles of incidence.

2 Traveling waves for Benney-Luke equations

The Benney-Luke equations that we consider have the form

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \varepsilon(\Phi_t\Delta\Phi + (\nabla\Phi)^2_t) = 0. \tag{2.1}$$

The variable $\Phi(x, y, t)$ is the nondimensional velocity potential on the bottom fluid boundary, and μ and ε are small parameters: $\sqrt{\mu} = h/L$ is the ratio of undisturbed fluid depth to typical wave length, and ε is the ratio of typical wave amplitude to fluid depth. Eq. (2.1) was derived for water waves without surface tension in [2] with $\mu = \varepsilon$, $a = \frac{1}{6}$ and $b = \frac{1}{2}$. As discussed in [14], Eq. (2.1) remains a formally valid water wave model in the presence of surface tension provided $a - b = \mathrm{Bo} - \frac{1}{3}$ where Bo is the Bond number (also see [12]). We take a and b to be positive for linear well-posedness. The surface elevation is related to Φ by $\eta = -\Phi_t + O(\mu, \varepsilon)$ to leading order.

For a traveling wave solution $\Phi = u(x - ct, y)$ of (2.1) the wave profile u should satisfy

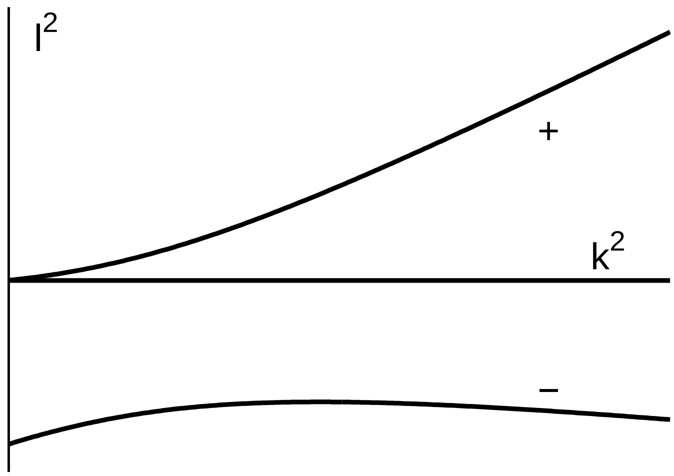
$$(c^2 - 1)u_{xx} - u_{yy} - \mu bc^2\Delta u_{xx} + \mu a\Delta^2 u - \varepsilon c(u_x\Delta u + |\nabla u|^2_x) = 0. \tag{2.2}$$

When the wave speed satisfies $0 < c^2 < \min(1, a/b)$ the traveling-wave equation (2.2) is of elliptic type. In this regime there exist finite-energy solitary waves or "lumps," as we proved in [14] using a variational method.

Here we will study the complementary case $c^2 > \max(1, a/b)$, meaning simply $c^2 > 1$ if $\mathrm{Bo} < \frac{1}{3}$, which is physically the more interesting case. In this regime the traveling-wave equation (2.2) has mixed type. Consider the linear dispersion relation for (2.2) for solutions of the form $u(x, y) = \exp(ikx + ily)$ with $\varepsilon = 0$:

$$-c^2k^2 + (k^2 + l^2)(1 - \mu bc^2k^2) + \mu a(k^2 + l^2)^2 = 0. \tag{2.3}$$

Figure 1: Schematic plot of the branches of the dispersion relation in (2.4)



Solving the quadratic for l^2 yields

$$l^2 = \pm\sqrt{q(k) + p(k)^2} - p(k) \tag{2.4}$$

where

$$p(k) = \frac{1}{2}\left(\frac{1}{\mu a} + \frac{2a - bc^2}{a}k^2\right), \qquad q(k) = \left(\frac{bc^2 - a}{a}\right)k^4 + \left(\frac{c^2 - 1}{\mu a}\right)k^2. \tag{2.5}$$

Since $bc^2 - a > 0$ and $c^2 - 1 > 0$, for all real k this yields four frequencies l with exactly two real and two purely imaginary. (See figure 1.) Moreover, $|l| \to \infty$ as $|k| \to \infty$ in an asymptotically linear fashion. The existence of unbounded imaginary branches of the linear dispersion relation indicates that the initial-value problem for (2.2) considered as an evolution equation in y is linearly *ill-posed*.

However, there is a spectral gap. It is evident from (2.4) with the minus sign that the imaginary branches of the dispersion relation are bounded away from the real axis — on these branches l^2 is strictly negative and $|\Im l| \geq c_0 > 0$ independent of k. Linear modes can grow or decay or remain neutral, but modes that grow or decay do so with rates bounded away from zero. This linear gap structure suggests that an invariant center manifold will exist in the nonlinear problem (2.2) with $\varepsilon > 0$. On such a manifold, modes that grow or decay in y will be slaved to the neutral modes in the infinite-dimensional space corresponding to real branches of the dispersion relation.

The main result of this paper involves proving that such an invariant center manifold of infinite dimension and codimension exists and contains all globally defined small-amplitude solutions of (2.2) that are periodic in the direction of propagation. These solutions are determined by suitable initial data along the line $y = 0$. In particular, we show that a complete traveling wave solution of (2.1) is determined uniquely by arbitrarily specifying along $y = 0$ the horizontal velocity (Φ_x, Φ_y) at the fluid bottom, provided these data are sufficiently small in an appropriate norm (the H^1 Sobolev norm). The bottom-velocity profile evolves in y according to a dispersive, nonlinear, nonlocal wave equation obtained by restriction to the invariant center manifold.

We obtain analogous results if we consider x as the time-like variable instead of y and specify initial data along $x = 0$, and we will only sketch the analysis in this case. The

traveling wave equation (2.2) is not isotropic, however, and it does not seem to be feasible in general to consider spatial dynamics in directions other than parallel and perpendicular to the direction of propagation of the wave.

Our bottom-velocity characterization of traveling waves for (2.1) is very similar to the wave-slope characterizations of traveling waves for the KP-II equation and other models that were discussed in [8]. For long waves of small amplitude, the variables involved are equivalent — to leading order, the wave slope is directly proportional to the time derivative of bottom velocity.

It would be desirable to describe the structure of the traveling wave solutions we find in some precise way as nonlinear superpositions of obliquely propagating waves — the waves exist with arbitrarily large translation velocity c, reminiscent of the "scissoring" behavior of superposed wave trains in the linear wave equation. Here we do not describe the wave structure with any accuracy beyond parametrization and linear approximation. But a more detailed study of the nonlinear equation governing evolution on the center manifold may reveal further structural information. Perhaps doubly periodic solutions can be found by finding y-periodic solutions of the governing nonlinear nonlocal wave equation on the center manifold, for example. We do not expect the waves to be doubly periodic in general, however.

The ill-posed mixed-type linear structure for the Benney-Luke equations considered here is like that for the exact water wave equations without surface tension [8]. In this respect the Benney-Luke equations are a faithful model for the exact water wave equations. The problem of finding a center manifold for these systems presents technical difficulties that have been little addressed in the literature. Both the center subspace (spanned by the neutral modes) and its complement (spanned by growing and decaying slave modes) are infinite-dimensional, and in both subspaces the evolution spectrum is unbounded. The structure resembles a wave equation coupled nonlinearly to an elliptic PDE. We are aware of only one work that concerns such a mixed-type problem, a paper of Mielke [11]. Our present problem is not of the special form that Mielke considered, but we can use some similar techniques to treat the peculiar difficulties that arise. In an appendix we present an abstract local theorem for center manifolds of infinite dimension and codimension which we can apply to obtain solutions of (2.2).

Ill-posed spatial evolution equations with finite-dimensional center manifolds arise for the exact water wave problem with surface tension and have recently been analyzed by Groves and Mielke [4, 5]. In the exact water wave problem without surface tension, Haragus and Pego [8] gave a linear analysis and identified some formally conserved quantities for spatial dynamics in the nonlinear problem. But the technical obstacles to proving there is a center manifold of infinite dimension and codimension in this case remain formidable.

3 Main Result

First, we convert the traveling wave equation (2.2) into a first order abstract differential equation in a Hilbert space H for which the "time-like" variable is y. Since we assume $a > 0$, $b > 0$ and $c^2 > \max(1, a/b)$, we can define positive numbers s, r, and d via

$$s^2 = \frac{bc^2 - a}{a}, \quad r^2 = \frac{c^2 - 1}{\mu a}, \quad d^2 = \frac{1}{\mu a}. \tag{3.1}$$

Introducing the variables U_1, U_2, U_3 and U_4 by

$$U_1 = u_x, \quad U_2 = u_y, \quad U_3 = u_{yy}, \quad U_4 = u_{yyy},$$

Eq. (2.2) can be viewed as the first order system

$$\frac{dU}{dy} = AU + f(U) \tag{3.2}$$

where

$$A = \begin{pmatrix} 0 & \partial_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ s^2\partial_x^3 - r^2\partial_x & 0 & (s^2-1)\partial_x^2 + d^2 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix},$$

$$f(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varepsilon c d^2 \left(3U_1\,\partial_x U_1 + U_1\,U_3 + 2U_2\,\partial_x U_2\right) \end{pmatrix}.$$

We will restrict ourselves to consider only solutions periodic in x with fixed period, which we take to be 2π by rescaling. By scaling amplitude we can assume $\varepsilon = 1$. We shall keep the parameters a, b, c and μ fixed.

Given an integer $k \geq 0$, let $\tilde{H}^k$ denote the Sobolev space of 2π-periodic functions on $\mathbb{R}$ whose weak derivatives up to order k are square-integrable. Then $\tilde{H}^k$ is a Hilbert space with norm given by

$$\|u\|^2_{\tilde{H}^k} = \sum_{j=0}^{k} \int_0^{2\pi} |\partial_x^j u|^2\, dx$$

To study (3.2) we introduce Hilbert spaces H and X defined by

$$H = \tilde{H}^1 \times \tilde{H}^1 \times \tilde{H}^0 \times \tilde{H}^{-1}, \tag{3.3}$$

$$X = \tilde{H}^2 \times \tilde{H}^2 \times \tilde{H}^1 \times \tilde{H}^0. \tag{3.4}$$

Note that X is densely embedded in H.

Our main result is the following.

Theorem 3.1 *(Traveling wave solutions via dynamics in y) There are positive constants δ_1 and C_1 with the following property: Given any initial conditions of the form*

$$(U_1(0), U_2(0)) = (w_1, w_2)$$

in $\tilde{H}^2 \times \tilde{H}^2$ such that $\|(w_1, w_2)\|_{\tilde{H}^1 \times \tilde{H}^1} \leq \delta_1$, Eq. (3.2) has a unique global classical solution $U \in C^1(\mathbb{R}, H) \cap C(\mathbb{R}, X)$ such that $\|U(y)\|_H \leq C_1\delta_1$ for all $y \in \mathbb{R}$. The map taking initial conditions to the solution is Lipschitz continuous from $\tilde{H}^1 \times \tilde{H}^1$ to $C([-T, T], H)$, for any $T > 0$.

Moreover, the first two components of the solution satisfy a dispersive, nonlinear, nonlocal wave equation of the form

$$\frac{d}{dy}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \\ S\partial_x & 0 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(U_1, U_2) \end{pmatrix}, \tag{3.5}$$

in which the map $g\colon \tilde{H}^1 \times \tilde{H}^1 \to \tilde{H}^1$ is Lipschitz with $g(0) = 0$, $Dg(0,0) = 0$, and where the nonlocal linear operator S (defined precisely in (6.11)) yields the real linear dispersion relation given by (2.4) with the plus sign.

It is evident from the stability estimates that, for any small initial data (w_1, w_2) in $\tilde{H}^1 \times \tilde{H}^1$, (3.2) has a global weak solution $U \in C(\mathbb{R}, H)$ which satisfies the same stability estimates.

For traveling wave solutions via dynamics in x, the roles of x and y are simply interchanged throughout. With $U = (u_y, u_x, u_{xx}, u_{xxx})$ one obtains a first order system of the same form as in (3.2) except that

$$s^2 = \frac{a}{bc^2 - a}, \quad r^2 = \frac{1}{\mu(bc^2 - a)}, \quad d^2 = \frac{c^2 - 1}{\mu(bc^2 - a)}. \tag{3.6}$$

and $f_4(U) = -\varepsilon c r^2(3U_2U_3 + U_2\partial_y U_1 + 2U_1\partial_y U_2)$. Exactly the same theorem as 3.1 holds in this case, with y replaced by x.

To prove these results, we will first apply the abstract local center manifold theorem in the appendix to our problem (see section 3). This will show that with respect to an invariant-subspace decomposition $H = H_0 \oplus H_1$ of H into a center subspace H_0 and a hyperbolic subspace H_1, equation (3.2) has a local invariant manifold given by the graph of a Lipschitz function $\phi_\delta\colon H_0 \to H_1 \cap X$.

Next, we will use a conserved energy functional $\mathcal{E}$ to prove that the zero solution is stable on the local center manifold, and infer the global existence of classical solutions for small data (see section 4). The energy functional is indefinite in general, but on the center subspace and center manifold it is coercive with respect to the H norm.

Finally, we will show in section 5 that the center subspace and center manifold are conveniently parametrized by the first two components of U, which correspond physically to the horizontal velocity on the fluid bottom. We will then obtain global classical solutions for (3.2) with data (w_1, w_2) in the space $\tilde{H}^2 \times \tilde{H}^2$ that are small in the $\tilde{H}^1 \times \tilde{H}^1$ norm, and verify that solutions satisfy an equation of the form in (3.5).

4 Existence of a local center manifold

The goal in this section is to verify the hypotheses of Theorem A.1 in the appendix (an abstract center manifold theorem) to obtain existence of a local center manifold for the system (3.2). These hypotheses come in three groups: (1) basic structural conditions on A and f, (2) existence of a spectral decomposition of H as a direct sum of a center subspace H_0 and a hyperbolic subspace H_1, with a corresponding decomposition of X, and (3) solvability conditions on linear equations of evolution in each subspace.

4.1 Structural conditions

First consider the basic structural conditions. It is straightforward to check that $A \in \mathcal{L}(X, H)$, the space of bounded linear operators from X to H. The map f is bilinear, and with $U, V \in H$ it is easy to check that

$$\|f(U+V) - f(U)\|_X \leq (\|U\|_H + \|V\|_H)\|V\|_H. \tag{4.1}$$

By consequence, $f\colon H \to X$ is smooth, and clearly $f(0) = 0 = Df(0)$. It is an interesting feature of this problem that the nonlinear map f exhibits a gain of regularity. We will exploit this feature to help find classical solutions.

4.2 Spectral decomposition

We will construct the desired spectral decompositions of H and X by using a complete set of eigenfunctions found via Fourier transform.

Any element $U \in H$ or X can be represented by a Fourier series

$$U = \sum_{k\in\mathbb{Z}} \hat{U}(k)e^{ikx}. \tag{4.2}$$

In terms of the vector Fourier coefficients the norms in H and X may be given by

$$\|U\|_H^2 = \sum_{k\in\mathbb{Z}} |S_H(k)\hat{U}(k)|^2, \qquad \|U\|_X^2 = \sum_{k\in\mathbb{Z}} (1+k^2)|S_H(k)\hat{U}(k)|^2, \tag{4.3}$$

where

$$S_H(k) = \operatorname{diag}\left\{(1+k^2)^{1/2}, (1+k^2)^{1/2}, 1, (1+k^2)^{-1/2}\right\}.$$

For $U \in X$ we have $\widehat{AU}(k) = \hat{A}(k)\hat{U}(k)$ where

$$\hat{A}(k) = \begin{pmatrix} 0 & ik & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -i(s^2k^3 + r^2k) & 0 & (1-s^2)k^2 + d^2 & 0 \end{pmatrix}.$$

If $\lambda \in \mathbb{C}$ is an eigenvalue with eigenfunction U, it must be that $(\hat{A}(k) - \lambda I)\hat{U}(k) = 0$ for all k, so that for some k, $\hat{U}(k)$ is an eigenvector of $\hat{A}(k)$ with eigenvalue λ. Let us write

$$Q(\lambda, k) = \det(\hat{A}(k) - \lambda I) = \lambda^4 - 2p(k)\lambda^2 - q(k) \tag{4.4}$$

where $p(k)$ and $q(k)$ are given in (2.5). An eigenvalue λ must satisfy $Q(\lambda, k) = 0$ for some k. For each nonzero integer k we find four distinct eigenvalues, two purely imaginary and two real, which we denote as follows:

$$\begin{aligned} \lambda_1(k) &= i\sqrt{-p(k) + \sqrt{p(k)^2 + q(k)}}, \quad && \lambda_2(k) = -\lambda_1(k), \\ \lambda_3(k) &= \sqrt{p(k) + \sqrt{p(k)^2 + q(k)}}, && \lambda_4(k) = -\lambda_3(k), \end{aligned} \tag{4.5}$$

For $k = 0$ the eigenvalue $\lambda_1(0) = \lambda_2(0) = 0$ is a double zero of $Q(\lambda, 0)$, and $\lambda_3(0) = -\lambda_4(0) = d$. As $k \to \infty$ we have that $\lambda_1(k) \sim isk$, $\lambda_3(k) \sim k$. The sign of $1 - s^2$ is not determined, but $\lambda_3(k) \geq \alpha > 0$ for some constant α independent of k. For each nonzero eigenvalue λ note that

$$\partial_\lambda Q(\lambda, k) = 4\lambda^3 - 4p(k)\lambda = \frac{2}{\lambda}\left(\lambda^4 + q(k)\right) \neq 0.$$

Corresponding to each nonzero eigenvalue $\lambda_m(k)$, the matrix $\hat{A}(k)$ has right eigenvector $v_m(k)$ and left eigenvector $w_m(k)$ given by

$$v_m(k) = \left(ik, \lambda_m(k), \lambda_m(k)^2, \lambda_m(k)^3\right)^T, \tag{4.6}$$

$$w_m(k) = \left(\frac{q(k)}{ik\lambda_m(k)}, \frac{q(k)}{\lambda_m(k)^2}, \lambda_m(k), 1\right) \cdot \frac{1}{\partial_\lambda Q(\lambda_m(k), k)}. \tag{4.7}$$

For the zero eigenvalue with $k = 0$ there is a two-dimensional eigenspace and we let

$$\begin{aligned} v_1(0) &= (1, 0, 0, 0)^T, \quad & v_2(0) &= (0, 1, 0, 0)^T, \\ w_1(0) &= (1, 0, 0, 0), & w_2(0) &= \left(0, 1, 0, -d^{-2}\right). \end{aligned} \tag{4.8}$$

The eigenvectors are normalized so that $w_m(k) \cdot v_m(k) = 1$. Introducing the matrices

$$W(k) = \begin{pmatrix} w_1(k) \\ w_2(k) \\ w_3(k) \\ w_4(k) \end{pmatrix}, \quad V(k) = (\, v_1(k), v_2(k), v_3(k), v_4(k)\,), \tag{4.9}$$

we have $W(k) \cdot V(k) = I$ and $W(k)\hat{A}(k)V(k) = \operatorname{diag}\{\lambda_1(k), \lambda_2(k), \lambda_3(k), \lambda_4(k)\}$ for all k.

Given an element U in H may write

$$\hat{U}(k) = V(k)U^\#(k) \qquad \text{where} \;\; U^\#(k) = \begin{pmatrix} U_1^\#(k) \\ U_2^\#(k) \\ U_3^\#(k) \\ U_4^\#(k) \end{pmatrix} = W(k) \cdot \hat{U}(k), \tag{4.10}$$

and hence we have the representations

$$U = \sum_{k\in\mathbb{Z}} \sum_{m=1}^{4} e^{ikx} v_m(k) U_m^\#(k), \qquad AU = \sum_{k\in\mathbb{Z}} \sum_{m=1}^{4} e^{ikx} v_m(k) U_m^\#(k) \lambda_m(k). \tag{4.11}$$

Because $|\lambda_m(k)|$ grows asymptotically linearly in k, it is not difficult to check that in terms of the coefficient vectors $U^\#(k)$ we have the following equivalences of norms:

$$\|U\|_H^2 \sim \sum_{k\in\mathbb{Z}} (1+k^2)^2 |U^\#(k)|^2, \qquad \|U\|_X^2 \sim \sum_{k\in\mathbb{Z}} (1+k^2)^3 |U^\#(k)|^2. \tag{4.12}$$

We define the projections π_0 and π_1 by

$$\pi_0 U = \sum_{k\in\mathbb{Z}} \sum_{m=1}^{2} e^{ikx} v_m(k) U_m^\#(k), \qquad \pi_1 U = \sum_{k\in\mathbb{Z}} \sum_{m=3}^{4} e^{ikx} v_m(k) U_m^\#(k). \tag{4.13}$$

From the equivalences in (4.12) it is evident that π_0 and π_1 are bounded on H and on X with $\pi_0 + \pi_1 = I$, and it is clear that $AX_j \subset H_j$ where $X_j = \pi_j X$ and $H_j = \pi_j H$ for $j = 0, 1$. This yields the spectral decompositions $H = H_0 \oplus H_1$ and $X = X_0 \oplus X_1$ with the properties required in the appendix.

4.3 Solvability conditions for linear dynamics.

First we consider the center subspace H_0. We define a family of linear operators $\{S_0(t)\}_{t\in\mathbb{R}}$ on H_0 by

$$S_0(t)U = \sum_{k\in\mathbb{Z}} \sum_{m=1}^{2} e^{ikx} v_m(k) U_m^\#(k) e^{\lambda_m(k)t}. \tag{4.14}$$

Since for $m = 1$ and 2, $\lambda_m(k)$ is pure imaginary and its magnitude grows asymptotically linearly in k, it is straightforward to show that the family $\{S_0(t)\}_{t\in\mathbb{R}}$ is a bounded C^0-group on H_0 with infinitesimal generator $A_0 = A|_{X_0}$. This establishes that the hypothesis (H0) of the appendix holds.

Next we seek to verify condition (H1) in the hyperbolic subspace H_1. For consistency with the notation in the appendix we replace y by t in the rest of this section. We must consider the inhomogeneous linear equation

$$\frac{d}{dt} U(t) = A_1 U(t) + G(t) \tag{4.15}$$

where $A_1 = A|_{X_1}$. Recall that $\lambda_3(k) \geq \alpha > 0$ for all k. Let $0 \leq \beta < \alpha$ and let $G \in C(\mathbb{R}, X_1) \cap H_1^\beta$, where as in the appendix for any Banach space Y we let

$$Y^\beta = \{u \in C(\mathbb{R}, Y) \mid \|u\|_{Y^\beta} := \sup_t e^{-\beta|t|} \|u\|_Y < \infty\}. \tag{4.16}$$

Suppose $U \in C^1(\mathbb{R}, H_1) \cap C(\mathbb{R}, X_1)$ is a solution belonging to H_1^β. Then applying the Fourier transform in x and multiplying by the matrix $W(k)$ yields

$$\frac{d}{dt} U_m^\#(k,t) = \lambda_m(k) U_m^\#(k,t) + G_m^\#(k,t) \tag{4.17}$$

for all $k \in \mathbb{Z}$, $t \in \mathbb{R}$ and $m = 3, 4$. The functions $G_m^\#(k,\cdot)$ and $U_m^\#(k,\cdot)$ belong to $\mathbb{R}^\beta$ ($Y = \mathbb{R}$ in (4.16)). Since $|\lambda_m(k)| \geq \alpha > \beta$, it is necessarily the case that

$$U_3^\#(k,t) = \int_\infty^t e^{\lambda_3(k)(t-\tau)} G_3^\#(k,\tau)\, d\tau, \qquad U_4^\#(k,t) = \int_{-\infty}^t e^{\lambda_4(k)(t-\tau)} G_4^\#(k,\tau)\, d\tau. \tag{4.18}$$

Consequently any solution of (4.15) in H_1^β is unique. We claim that the formulas (4.18) together with the representation for $U = \pi_1 U$ in (4.13) yield existence of a solution in H_β^1. For this purpose it will be convenient to decompose Eq. (4.15) using projections into the "unstable" and "stable" subspaces. These projections are defined by

$$\pi_u U = \sum_{k\in\mathbb{Z}} e^{ikx} v_3(k) U_3^\#(k), \qquad \pi_s U = \sum_{k\in\mathbb{Z}} e^{ikx} v_4(k) U_4^\#(k) \tag{4.19}$$

for $U \in H$. Clearly π_u and π_s are bounded on H and X and $\pi_u + \pi_s = \pi_1$.

We next introduce a Green's function operator $S(t)$ defined for nonzero $t \in \mathbb{R}$ by

$$S(t)U = \begin{cases} -\displaystyle\sum_{k\in\mathbb{Z}} e^{ikx} v_3(k) U_3^\#(k) e^{\lambda_3(k)t} & \text{for } t < 0, \\ \displaystyle\sum_{k\in\mathbb{Z}} e^{ikx} v_4(k) U_4^\#(k) e^{\lambda_4(k)t} & \text{for } t > 0. \end{cases} \tag{4.20}$$

Due to (4.12), $S(t)$ is equivalent to a multiplication operator on a sequence space ℓ^2, so it is easy to verify that for some constant C independent of t we have the following norm bounds:

$$\|S(t)\|_{\mathcal{L}(Y)} \le Ce^{-\alpha|t|} \qquad (Y = H \text{ or } X), \tag{4.21}$$

$$\|S(t)\|_{\mathcal{L}(H,X)} \le \begin{cases} Ce^{-\alpha t} & \text{for } \alpha|t| \ge 1, \\ C|t|^{-1} & \text{for } \alpha|t| \le 1, \end{cases} \tag{4.22}$$

$$\|S(t) - \pi_s\|_{\mathcal{L}(X,H)} + \|S(-t) + \pi_u\|_{\mathcal{L}(X,H)} \le Ct \quad \text{for } t > 0. \tag{4.23}$$

Here $\mathcal{L}(Y)$ and $\mathcal{L}(H, X)$ respectively denote the space of bounded operators on Y, and from H to X. To get these bounds one uses the facts that $\lambda_3(k) = -\lambda_4(k)$ is bounded below by $\alpha > 0$ and grows linearly in k, so

$$\|\pi_1 U\|_X^2 \sim \sum_{k\in\mathbb{Z}} \sum_{m=3}^{4} (1 + k^2)^2 |\lambda_m(k) U_m^\#(k)|^2,$$

together with the facts that for $t > 0$,

$$\sup_{\lambda\ge\alpha} \lambda^{-1}|e^{-\lambda t} - 1| \le t, \qquad \sup_{\lambda\ge\alpha} \lambda e^{-\lambda t} = \begin{cases} \alpha e^{-\alpha t} & \text{for } \alpha t \ge 1, \\ 1/et & \text{for } \alpha t \le 1. \end{cases}$$

Also, S is C^1 from $\mathbb{R} \setminus \{0\}$ to $\mathcal{L}(H)$ with $dS(t)/dt = A_1 S(t)$, and $S(t) \to \pi_s$ (resp. $-\pi_u$) strongly as $t \to 0^+$ (resp. 0^-). Therefore the families $\{S(t)\}_{t>0}$ and $\{-S(-t)\}_{t>0}$ are analytic semigroups in $\pi_s H$ and $\pi_u H$ respectively [13, p.62].

Eqs. (4.18) yield the formula

$$U(t) = \int_{-\infty}^{\infty} S(t-\tau) G(\tau)\, d\tau \tag{4.24}$$

for the solution of (4.15). We wish to show that $U \in C(\mathbb{R}, X)$, $U \in H_1^\beta$, and dU/dt exists in H and satisfies (4.15). For the first step we note that

$$U(t) = \int_{|s|\le\alpha^{-1}} S(s) G(t-s)\, ds + \int_{|s|\ge\alpha^{-1}} S(s) G(t-s)\, ds.$$

Using (4.21) and $G \in C(\mathbb{R}, X)$ it is clear that the first term is in $C(\mathbb{R}, X)$. For the second term we use (4.22) and $G \in H^\beta$ to see that the integral converges in X uniformly on compact sets in t.

It follows that $U \in C(\mathbb{R}, X)$. Using (4.21) we have that for $Y = H$ or X, if $G \in Y^\beta$ then

$$e^{-\beta|t|}\|U(t)\|_Y \le C\|G\|_{Y^\beta} \int_{-\infty}^{\infty} e^{-\alpha|s|+\beta(|t-s|-|t|)}\, ds \le \frac{2C}{\alpha-\beta}\|G\|_{Y^\beta}. \tag{4.25}$$

This shows that $U \in Y^\beta$ and establishes the estimates required in hypothesis (H1).

It remains to show $U = \pi_s U + \pi_u U$ is differentiable in H and satisfies (4.15). We check this for the two terms separately. For $h > 0$ we compute

$$\begin{aligned} \frac{\pi_s U(t+h) - \pi_s U(t)}{h} &= \left(\frac{S(h)-\pi_s}{h}\right)\pi_s U(t) + \frac{1}{h}\int_0^h (S(\tau)-\pi_s)G(t+h-\tau)\, d\tau \\ &\quad + \frac{1}{h}\int_t^{t+h} \pi_s G(\tau)\, d\tau. \end{aligned}$$

Using (4.23) and $G \in C(\mathbb{R}, X)$, as $h \to 0^+$ we deduce that the next-to-last term converges to zero in H and the last term converges to $\pi_s G(t)$. Moreover the first term converges to $A_1 \pi_s U(t)$. Hence the right derivative exists and satisfies $D_+\pi_s U(t) = A_1\pi_s U(t) + \pi_s G(t)$, so is continuous into H. It follows that $\pi_s U$ is differentiable. We may treat $\pi_u U$ in a similar way, and conclude that U is differentiable and satisfies (4.15).

This completes the verification of hypothesis (H1) in the appendix. Since all the hypotheses of Theorem A.1 have been verified, we have established that system (3.2) admits a local center manifold having the properties stated in the Theorem.

5 Global existence and stability

Our aim in this section is to establish the global existence of classical solutions on the local center manifold, for initial data that is small in H-norm. This we do by establishing that the zero solution is stable on the center manifold, which is given by the graph of a function $\phi_\delta\colon H_0 \to X_1$. We shall exploit an energy functional that is conserved in "time" for classical solutions. We shall prove the following result.

Theorem 5.1 *(Stability on the center manifold) Let ϕ_δ be given by applying Theorem A.1 to (3.2). There exist positive constants δ_2 and C_2 such that, for any $\xi \in X_0$ with $\|\xi\|_H \le \delta_2$, there is a unique classical solution U on $\mathbb{R}$ to (3.2) such that $\pi_0 U(0) = \xi$ and $\|U(y)\|_H \le 2C_2\|\xi\|_H$ for all $y \in \mathbb{R}$. Moreover, for any $T > 0$ the map taking ξ to U is Lipschitz continuous from H_0 to $C([-T,T], H)$.*

We define the functional $\mathcal{E}\colon H \to \mathbb{R}$ by

$$\mathcal{E}(U) = \mathcal{E}_0(U) + \mathcal{E}_1(U) \tag{5.1}$$

where the quadratic part is

$$\begin{aligned} \mathcal{E}_0(U) \;&:=\; \frac{1}{2\pi}\int_0^{2\pi} \Big(r^2|U_1|^2 + s^2|\partial_x U_1|^2 + d^2|U_2|^2 - (s^2-1)|\partial_x U_2|^2 + |U_3|^2 \Big)\, dx \\ &\quad - \frac{1}{\pi}\,(U_4, U_2)_{-1,1}\,, \end{aligned} \tag{5.2}$$

and the remaining part is

$$\mathcal{E}_1(U) = -\frac{\varepsilon c d^2}{2\pi}\int_0^{2\pi} (U_1^3 - U_1 U_2^2)\, dx. \tag{5.3}$$

Above, $(\cdot,\cdot)_{-1,1}$ denotes the natural pairing between $\tilde{H}^{-1}$ and $\tilde{H}^{1}$. Clearly $\mathcal{E}$ is smooth from H to $\mathbb{R}$. If $U \in C^1(\mathbb{R}, H)$ is a classical solution of the first order equation (3.2), then for all $y \in \mathbb{R}$ we find

$$\frac{d}{dy}\mathcal{E}(U(y)) = 0.$$

In other words, $\mathcal{E}$ is conserved along classical solutions of (3.2).

In the case of dynamics in x, when x and y are interchanged, the quadratic part of the energy has the same form and the remaining part is replaced by

$$\mathcal{E}_1(U) = \frac{-\varepsilon c r^2}{\pi}\int_0^{2\pi} U_2^3\,dy. \tag{5.4}$$

Note that neither $\mathcal{E}$ nor $\mathcal{E}_0$ is necessarily positive. However, we will establish that their restrictions to the center manifold are positive. We will first show that $\mathcal{E}_0$ is positive in the center space H_0.

Lemma 5.2 *There is a positive constant C_0 such that for any $U \in H_0$,*

$$(1/C_0)\,\|U\|_H^2 \le \mathcal{E}_0(U) \le C_0\|U\|_H^2.$$

Proof. Using the Fourier series representation (4.2) we find

$$\mathcal{E}_0(U) = \sum_{k\in\mathbb{Z}} \mathcal{E}_0(\hat{U}(k)e^{ikx}). \tag{5.5}$$

Now

$$\hat{U}(0) = \begin{pmatrix} U_1^{\#}(0) \\ U_2^{\#}(0) \\ 0 \\ 0 \end{pmatrix}, \quad \hat{U}(k) = \begin{pmatrix} ik(U_1^{\#}(k) + U_2^{\#}(k)) \\ \lambda_1(k)(U_1^{\#}(k) - U_2^{\#}(k)) \\ \lambda_1(k)^2(U_1^{\#}(k) + U_2^{\#}(k)) \\ \lambda_1(k)^3(U_1^{\#}(k) - U_2^{\#}(k)) \end{pmatrix}, \tag{5.6}$$

so

$$\mathcal{E}_0(\hat{U}(0)) = r^2|U_1^{\#}(0)|^2 + d^2|U_2^{\#}(0)|^2,$$

and since $\lambda_1(k)$ is purely imaginary we compute for $k \neq 0$ that

$$\begin{aligned} \mathcal{E}_0(\hat{U}(k)e^{ikx}) &= (r^2 + s^2k^2)|\hat{U}_1(k)|^2 + (d^2 + (1-s^2)k^2)|\hat{U}_2(k)|^2 + |\hat{U}_3(k)|^2 - 2\hat{U}_4(k)\overline{\hat{U}_2(k)} \\ &= (q(k) + |\lambda_1(k)|^4)|U_1^{\#}(k) + U_2^{\#}(k)|^2 \\ &\quad + |\lambda_1(k)|^2(2p(k) - 2\lambda_1(k)^2)|U_1^{\#}(k) - U_2^{\#}(k)|^2. \end{aligned} \tag{5.7}$$

The quantity

$$L(k) := 2p(k) - 2\lambda_1(k)^2 = 2\sqrt{p(k)^2 + q(k)} \tag{5.8}$$

is positive, and for $|k| \ge 1$, the quantity $|\lambda_1(k)|^2 L(k)$ is bounded above and below by a constant times $(1+k^2)^2$. The same is true for $q(k) + |\lambda_1(k)|^4$. Therefore by (4.12) we obtain the desired equivalence:

$$\begin{aligned} \mathcal{E}_0(U) &\sim \sum_{k\in\mathbb{Z}}(1+k^2)^2\left(|U_1^{\#}(k) + U_2^{\#}(k)|^2 + |U_1^{\#}(k) - U_2^{\#}(k)|^2\right) \\ &= 2\sum_{k\in\mathbb{Z}}(1+k^2)^2\left(|U_1^{\#}(k)|^2 + |U_2^{\#}(k)|^2\right) \sim \|U\|_H^2. \qquad \square \end{aligned} \tag{5.9}$$

Next we control the energy on the center manifold, which is given by the graph of a function $\phi_\delta\colon H_0 \to X_1$ with the properties stated in Theorem A.1.

Lemma 5.3 *Let ϕ_δ be given by applying Theorem A.1 to (3.2). Then there exist positive constants δ_2 and C_2 such that for all $\xi \in H_0$ with $\|\xi\|_H < \delta_2$ we have*

$$\frac{1}{C_2}\|\xi\|_H^2 \le \mathcal{E}(\xi + \phi_\delta(\xi)) \le C_2\|\xi\|_H^2.$$

Proof. By Theorem A.1, $\|\phi_\delta(\xi)\|_H = o(\|\xi\|_H)$ as $\|\xi\|_H \to 0$, and since $\mathcal{E}_1(\xi + \phi_\delta(\xi)) = O(\|\xi\|_H^3)$ it is easy to see that

$$\begin{aligned} \mathcal{E}(\xi + \phi_\delta(\xi)) &= \mathcal{E}_0(\xi) + O(\|\xi\|_H\|\phi_\delta(\xi)\|_H) + \mathcal{E}_1(\xi + \phi_\delta(\xi)) && (5.10) \\ &= \mathcal{E}_0(\xi) + o(\|\xi\|_H^2) && (5.11) \end{aligned}$$

as $\|\xi\|_H \to 0$. The result follows using Lemma 5.2. $\square$

Let us now proceed to prove Theorem 5.1. Let δ_2 and C_2 be given by Lemma 5.3. We may assume $2C_2\delta_2 < \delta$ by making δ_2 smaller if necessary. Suppose $\xi \in X_0$ with $\|\xi\|_H \le \delta_2$. We invoke Theorem A.1 and obtain existence of a continuous function U from $\mathbb{R}$ into the center manifold $\mathcal{M}_\delta = \{\zeta + \phi_\delta(\zeta) : \zeta \in X_0\}$ such that $\pi_0 U(0) = \xi$, which is a classical solution of (3.2) on any open interval $J \subset \mathbb{R}$ containing 0 such that $\|\pi_0 U(y)\|_H < \delta$ for all $y \in J$.

By a straightforward continuation argument, to show that U is a classical solution on $\mathbb{R}$ and satisfies the estimate claimed in Theorem 5.1, it suffices to establish an appropriate a priori estimate. Namely, it is enough to show that on any open interval $J \subset \mathbb{R}$ containing 0 such that $\|\pi_0 U(y)\|_H < \delta$ for all $y \in J$, we have that $\|\pi_0 U(y)\|_H \le C_2\|\xi\|_H$ for all $y \in J$.

Since U is a classical solution on such an interval, we may use the fact that $\mathcal{E}(U(y))$ is constant along with Lemma 5.3 to deduce that for any $y \in J$,

$$\frac{1}{C_2}\|\pi_0 U(y)\|_H^2 \le \mathcal{E}(U(y)) = \mathcal{E}(U(0)) \le C_2\|\xi\|_H^2. \tag{5.12}$$

This establishes the desired a priori estimate, proving the existence part of the Theorem. The statement regarding Lipschitz dependence follows from Proposition A.6.

To prove the uniqueness statement, suppose U is a classical solution on $\mathbb{R}$ to (3.2) such that $\pi_0 U(0) = \xi$ and $\|U(y)\|_H \le 2C_2\|\xi\|_H$ for all $y \in \mathbb{R}$. Since $2C_2\|\xi\|_H < \delta$, from Theorem A.1(v) it follows $U(y)$ must lie on the center manifold $\mathcal{M}_\delta$ for all $y \in \mathbb{R}$. Then U is determined by $U_0 = \pi_0 U$, which by Theorem A.1(iii) is the unique classical solution of the equation

$$\frac{d}{dy}U_0(y) = A_0 U_0(y) + \pi_0 f(U_0(y) + \phi_\delta(U_0(y))) \tag{5.13}$$

with $U_0(0) = \xi$.

6 Parametrization and evolution on the center manifold

The last steps needed to prove Theorem 3.1 involve showing that global solutions U on the center manifold can be characterized by their initial data in just the first two components. In order to accomplish this characterization, by Theorem 5.1 it will suffice to establish a suitable correspondence between the first two components of $U(0)$ and the center-subspace projection $\pi_0 U(0)$. In this section, we denote the restriction of any $U = (U_1, U_2, U_3, U_4)^T$ on the first two components by

$$\mathcal{R}U = (U_1, U_2). \tag{6.1}$$

Theorem 6.1 *Let ϕ_δ be given by applying Theorem A.1 to (3.2). There exist positive constants δ_3 and C_3 with the following property. For any $w = (w_1, w_2) \in \tilde{H}^1 \times \tilde{H}^1$ such that $\|w\|_{\tilde{H}^1\times\tilde{H}^1} < \delta_3$, there exists a unique $\xi \in H_0$ such that $\|\xi\|_H \le C_3\|w\|_{\tilde{H}^1\times\tilde{H}^1}$ and*

$$w = \mathcal{R}(\xi + \phi_\delta(\xi)). \tag{6.2}$$

The map $w \to \xi$ is Lipschitz continuous, and if $w \in \tilde{H}^2 \times \tilde{H}^2$ then $\xi \in X_0$.

To prove this, first we study the restriction $\mathcal{R}$ on the center subspace.

Lemma 6.2 *The map $\mathcal{R}$ yields simultaneous isomorphisms $H_0 \cong \tilde{H}^1 \times \tilde{H}^1$ and $X_0 \cong \tilde{H}^2 \times \tilde{H}^2$.*

Proof. Given any $U = \pi_0 U$ in H_0 or X_0, we use the representation (4.13) to write

$$\mathcal{R}U = \sum_{k\in\mathbb{Z}} e^{ikx}\tilde{V}(k)\begin{pmatrix} U_1^\#(k) \\ U_2^\#(k)\end{pmatrix}. \tag{6.3}$$

where

$$\tilde{V}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad \tilde{V}(k) = \begin{pmatrix} ik & ik \\ \lambda_1(k) & -\lambda_1(k)\end{pmatrix} \tag{6.4}$$

for $k \ne 0$. Since $|\lambda_1(k)|$ grows asymptotically linearly in k, it follows that

$$|\widehat{\mathcal{R}U}(k)|^2 \le C(1+k^2)(|U_1^\#(k)|^2 + |U_2^\#(k)|^2). \tag{6.5}$$

From the equivalences (4.12) it follows that $\mathcal{R}$ is bounded from H_0 to $\tilde{H}^1 \times \tilde{H}^1$ and from X_0 to $\tilde{H}^2 \times \tilde{H}^2$. Moreover, $\mathcal{R}$ is one-to-one, since if $\mathcal{R}U = 0$ then $\hat{U}(k) = 0$ for all k since $\tilde{V}(k)$ is invertible.

To see that $\mathcal{R}$ yields an isomorphism, we observe that its inverse is given by the prolongation formula $U = \mathcal{P}w$, where, given $w \in \tilde{H}^1 \times \tilde{H}^1$ or $\tilde{H}^2 \times \tilde{H}^2$, we have $U = \pi_0 U$ given by (4.13) with

$$\begin{pmatrix} U_1^\#(k) \\ U_2^\#(k)\end{pmatrix} = \tilde{V}(k)^{-1}\hat{w}(k) = \frac{1}{2ik\lambda_1(k)}\begin{pmatrix} \lambda_1(k) & ik \\ \lambda_1(k) & -ik\end{pmatrix}\begin{pmatrix} \hat{w}_1(k) \\ \hat{w}_2(k)\end{pmatrix} \tag{6.6}$$

for $k \ne 0$. Clearly

$$|U_1^\#(k)|^2 + |U_2^\#(k)|^2 = |\tilde{V}(k)^{-1}\hat{w}(k)|^2 \le \frac{C}{1+k^2}|\hat{w}(k)|^2$$

for all k, so using (4.12) we see that $\mathcal{P}$ is bounded from $\tilde{H}^1 \times \tilde{H}^1$ to H_0 and from $\tilde{H}^2 \times \tilde{H}^2$ to X_0. $\square$

Proof of Theorem 6.1. Given w, we shall prove corresponding statements for $\zeta = \mathcal{R}\xi$ in place of ξ and use Lemma 6.2. Recall that $\mathcal{P}$, the inverse of $\mathcal{R}$, is given by $U = \mathcal{P}w$ using (6.6) and (4.13). The quantity ζ should satisfy

$$\zeta = w - \tilde{\phi}(\zeta) \qquad \text{where} \quad \tilde{\phi}(\zeta) = \mathcal{R}\phi_\delta(\mathcal{P}\zeta). \tag{6.7}$$

Using the contraction mapping theorem, we find that this equation has a unique solution $\zeta \in \tilde{H}^1 \times \tilde{H}^1$ satisfying $\|\zeta\|_{\tilde{H}^1\times\tilde{H}^1} < \delta'$ if $\|w\|_{\tilde{H}^1\times\tilde{H}^1} < \delta'(1 - L(\delta'))$, provided that the Lipschitz constant $L(\delta')$ of $\tilde{\phi}$ on the ball of radius δ' in $\tilde{H}^1 \times \tilde{H}^1$ satisfies $L(\delta') < 1$. This is true if δ' is sufficiently small, due to Theorem A.1(ii) and Lemma 6.2. Moreover, the map $w \mapsto \zeta$ is Lipschitz with Lipschitz constant $1/(1 - L(\delta'))$. Furthermore, $\tilde{\phi}(\zeta) \in \mathcal{R}X_1 \subset \tilde{H}^2 \times \tilde{H}^2$, so if $w \in \tilde{H}^2 \times \tilde{H}^2$ then $\zeta \in \tilde{H}^2 \times \tilde{H}^2$ and so $\xi = \mathcal{P}\zeta \in X_0$. $\square$

Proof of Theorem 3.1. The parts of the Theorem referring to existence, uniqueness, stability, and Lipschitz dependence on initial data follow directly from Theorems 5.1 and 6.1. It remains to verify that the first two components $w = \mathcal{R}U$ of a solution as given by these results satisfy an equation of the form (3.5). Let $\xi(y)$ be determined from $w(y)$ by Theorem 6.1 for each y. Since $\mathcal{P}\mathcal{R}\xi = \xi$, we find that

$$U = \mathcal{P}w + \psi(w) \qquad \text{where} \quad \psi(w) = (I - \mathcal{P}\mathcal{R})\phi_\delta(\xi). \tag{6.8}$$

Note that ϕ_δ takes values in X_1 so $\psi(w)$ need not be zero. However the first two components $\mathcal{R}\psi(w) = 0$, and since U is a classical solution of (3.2), restriction to the first two components yields

$$\frac{d}{dy} w = \mathcal{R}A(\mathcal{P}w + \psi(w)) = \mathcal{A}w + \begin{pmatrix} 0 \\ \psi_3(w) \end{pmatrix} \tag{6.9}$$

since the first two components $\mathcal{R}f(U) = 0$. Here the action of the operator $\mathcal{A} = \mathcal{R}A\mathcal{P}$ can be determined through Fourier transform representation from (4.11) and (6.6). We find that

$$\begin{aligned} \mathcal{A}w &= \sum_{k\in\mathbb{Z}} e^{ikx} \tilde{V}(k) \begin{pmatrix} \lambda_1(k) & 0 \\ 0 & -\lambda_1(k) \end{pmatrix} \tilde{V}(k)^{-1} \hat{w}(k) \\ &= \sum_{k\neq 0} e^{ikx} \begin{pmatrix} 0 & ik \\ \lambda_1(k)^2/ik & 0 \end{pmatrix} \begin{pmatrix} \hat{w}_1(k) \\ \hat{w}_2(k) \end{pmatrix}. \end{aligned} \tag{6.10}$$

Thus we have that $w = \mathcal{R}U$ satisfies an equation of the form (3.5) in which $g = \psi_3$ and S is a pseudodifferential operator of degree zero defined by

$$\widehat{Sw_1}(k) = (\lambda_1(k)/ik)^2 \hat{w}_1(k) \tag{6.11}$$

for $k \neq 0$. The eigenvalues of $\mathcal{A}$ have the form $\pm\lambda_1(k)$, leading to the real dispersion relation in (2.4) with the plus sign. Since ϕ_δ takes values in X_1 and satisfies $\phi_\delta(0) = 0$, $D\phi_\delta(0) = 0$, we find that $g(w_1, w_2) = \psi_3(w)$ is Lipschitz from a small ball in $\tilde{H}^1 \times \tilde{H}^1$ into $\tilde{H}_1$ with $g(0,0) = 0$, $Dg(0,0) = 0$.

This finishes the proof of Theorem 3.1. $\square$

A Center manifolds of infinite dimension and codimension

Here we consider abstract differential equations of the form

$$\frac{du}{dt}(t) = Au(t) + f(u(t)). \tag{A.1}$$

where X and H are Banach spaces with X densely embedded in H, $A \in \mathcal{L}(X, H)$, the space of bounded linear operators from X to H, and f is continuously differentiable from H into X with $f(0) = 0$ and $Df(0) = 0$.

The goal in this section is to prove the existence of a locally invariant center manifold of classical solutions for the system (A.1) under certain conditions which permit the center subspace (that associated with the purely imaginary spectrum of A) to have infinite dimension and codimension.

We start with some basic definitions and some hypotheses:

Definition A.1 *Let $J \subset \mathbb{R}$ be an open interval and $u\colon \mathbb{R} \to H$ be a function. We say that u is a classical solution of (A.1) on J if the mapping $t \mapsto u(t)$ is continuous from J into X, is differentiable from J into H and (A.1) holds for all $t \in J$.*

Let $\beta > 0$, let Y and Z be Banach spaces and U be an open set in Y. We define the Banach spaces $C_b(U,Z)$, $\mathrm{Lip}(U,Z)$ and Y^β by

$$\begin{aligned}
C_b(U,Z) &:= \left\{ f \in C(U,Z) : \sup_{u\in Z} \|f(u)\|_Z < \infty \right\}. \\
\mathrm{Lip}(U,Z) &:= \{ f \in C(U,Z) : \|f(u)-f(v)\|_Z \le M_f \|u-v\|_Y \text{ for all } u,v \in U \}. \\
Y^\beta &:= \{ u \in C(\mathbb{R},Y) : ||u||_{Y^\beta} := \sup_t e^{-\beta|t|} \|u(t)\|_Y < \infty \}.
\end{aligned}$$

Throughout this section we assume that there are bounded projections π_0 and π_1 on H such that (i) $H = H_0 \oplus H_1$ with $H_i := \pi_i(H)$, (ii) $\pi_i|_X$ is bounded from X to X, and (iii) $AX_i \subseteq H_i$ where $X_i := \pi_i(X)$, for $i = 0,1$. We let c_π denote a common norm bound for the projections π_j on H and X, $j = 0,1$.

In consequence, the equation (A.1) can be rewritten as the first order system

$$\begin{aligned}
\frac{d}{dt}u_0(t) &= A_0 u_0(t) + \pi_0 f(u(t)), \\
\frac{d}{dt}u_1(t) &= A_1 u_1(t) + \pi_1 f(u(t)),
\end{aligned} \tag{A.2}$$

where $A_i \in \mathcal{L}(X_i, H_i)$ with $A_i y = \pi_i A y$ for $y \in X_i$.

We assume the following splitting properties for the operator A, associated with the linear evolution equation $du/dt = Au$.

(H0) A_0 is the generator of a C^0-group $\{S_0(t)\}_{t\in\mathbb{R}}$ on H_0 with subexponential growth. I.e., given any $\beta > 0$, there is a constant $M_0(\beta) > 0$ such that

$$\|S_0(t)\|_{\mathcal{L}(H_0)} \le M_0(\beta) e^{\beta|t|} \quad \text{for all } t \in \mathbb{R}.$$

(H1) There exists $\alpha > 0$ and a positive function M_1 on $[0,\alpha)$ such that for any $\beta \in [0,\alpha)$ and for any $g_1 \in C(\mathbb{R}, X_1) \cap H_1^\beta$ the equation

$$\frac{d}{dt}u_1 = A_1 u_1 + g_1 \tag{A.3}$$

has a unique solution in H_1^β given by $u_1 = K_1 g_1$, where $K_1 \in \mathcal{L}(H_1^\beta)$ with $\|K_1\|_{\mathcal{L}(H_1^\beta)} \le M_1(\beta)$. Furthermore $\|K_1\|_{\mathcal{L}(X_1^\beta)} \le M_1(\beta)$.

Theorem A.1 (Local Center Manifold Theorem) *Let H, X, A, π_0, π_1 and f be as above, and let*

$$B(\delta) = \{ y \in H_0 : \|y\|_H < \delta \}.$$

Then for all sufficiently small $\delta > 0$ there exists $\phi_\delta \colon H_0 \to X_1$ such that

(i) $\phi_\delta(0) = 0$ *and* $D\phi_\delta(0) = 0$.

(ii) $\phi_\delta \in C_b(H_0, X_1) \cap \mathrm{Lip}(H_0, X_1)$, *and on any ball $B(\delta')$, ϕ_δ has Lipschitz constant $L(\delta')$ satisfying $L(\delta') < \frac{1}{2}$ and $L(\delta') \to 0$ as $\delta' \to 0^+$.*

(iii) The manifold $\mathcal{M}_\delta \subset X$ given by

$$\mathcal{M}_\delta := \{ \xi + \phi_\delta(\xi) : \xi \in X_0 \} \tag{A.4}$$

is a local integral manifold for (A.1) over $B(\delta) \cap X_0$. That is, given any $y \in \mathcal{M}_\delta$ there is a continuous map $u \colon \mathbb{R} \to \mathcal{M}_\delta$ with $u(0) = y$, such that for any open interval J

containing 0 *with* $\pi_0 u(J) \subset B(\delta)$ *it follows that* u *is a classical solution of (A.1) on* J*. Moreover,* $u_0 := \pi_0 u$ *is the unique classical solution on* J *with* $u_0(0) = \pi_0 y$ *to the reduced equation*

$$\frac{d}{dt}u_0(t) = A_0 u_0(t) + F_\delta(u_0(t)), \tag{A.5}$$

where $F_\delta \colon H_0 \to X_0$ *is locally Lipschitz and is given by* $F_\delta(w) := \pi_0 f(w + \phi_\delta(w))$*.*

(iv) For any open interval $J \subset \mathbb{R}$*, every classical solution* $u_0 \in C^1(J, H_0) \cap C(J, X_0)$ *of the reduced equation (A.5) such that* $u_0(t) \in B(\delta)$ *for all* $t \in J$ *yields, via* $u = u_0 + \phi_\delta(u_0)$*, a classical solution* u *of the full equation (A.1) on* J*.*

(v) The manifold $\mathcal{M}_\delta$ *contains all classical solutions on* $\mathbb{R}$ *that satisfy* $\|u(t)\|_H \le \delta$ *for all* t*.*

Center manifolds of infinite dimension with finite codimension ($\dim H_1 < \infty$) were obtained by Bates and Jones [1]. A generalization regarding invariant manifolds of infinite dimension and codimension in nonautonomous systems was obtained by Scarpellini [15], but his hypotheses require that the operator A_1 be bounded from H to H. The general strategy of our proof will follow closely the lines of [16] (also see [9, 10, 11]) for the case of a finite-dimensional center manifold in an ill-posed system for which the spectrum of A_1 is unbounded on both sides of the imaginary axis. One transforms Eq. (A.1) into an integral equation that must contain all small bounded solutions. In order to obtain an invariant manifold by a contraction mapping argument, one must modify the nonlinearity f outside a neighborhood of 0 using a cutoff function.

A significant point of difference between our results and those of [16] is that our cutoff occurs in the H norm, and not in the X norm as in [16]. In our application to traveling waves of the Benney-Luke equation, it is important to establish global existence of small solutions on the center manifold. We do this by using an energy functional which is defined on H and conserved in time for classical solutions (which take values in X). The energy is indefinite in general, but controls the H norm for solutions on the center manifold. For this reason, we find it necessary to obtain a center manifold that contains solutions with large X norm but small H norm. This we accomplish by requiring that the nonlinearity f has a smoothing property, mapping H into X. This is a stronger hypothesis on f than is made in earlier works, but we want to emphasize that it is completely natural for the present application to the Benney-Luke equation.

A.1 Linear Analysis

In this section we discuss the existence of classical solutions for inhomogeneous linear problems that are associated with the system (A.2). First we begin with the linear analysis corresponding to the problem on the center space. The following result follows easily from the standard theory of C^0 semigroups (see [13]).

Lemma A.2 *(Linear analysis on the center space) Suppose condition (H0) holds. Then for any* $\zeta \in H_0$ *and* $g_0 \in H_0^\beta$ *the initial value problem*

$$\frac{d}{dt}u_0(t) = A_0 u_0(t) + g_0(t), \quad u(0) = \zeta, \tag{A.6}$$

has a global mild solution $u_0 \in H_0^\beta$ *given by*

$$u_0(t) = S_0(t)\zeta + \int_0^t S_0(t-\tau) g_0(\tau) d\tau. \tag{A.7}$$

Moreover, if $g_0 \in C(\mathbb{R}, X_0)$ then for every $\zeta \in X_0$ Eq. (A.6) has a unique global classical solution. In either case $Y = H_0$ or X_0, for $g_0 \in Y_\beta$ we have the estimate

$$||u_0||_{Y^\beta} \leq M_0(\beta)\|\zeta\|_Y + \frac{M_0(\beta/2)}{\beta/2}||g_0||_{Y^\beta}. \tag{A.8}$$

Now we can treat the full linear problem on H by simply combining the results obtained in the center space and the hypothesis (H1) in the hyperbolic space.

Lemma A.3 *(Combined linear analysis) Let H, X, A, π_0, π_1 be as above and let $\beta \in (0, \alpha)$ be fixed. Then for every $\zeta \in X_0$ and for every $g \in H^\beta \cap C(\mathbb{R}, X)$, the problem*

$$\frac{d}{dt}u(t) = Au(t) + g(t) \quad (t \in \mathbb{R}), \qquad \pi_0 u(0) = \zeta \tag{A.9}$$

has a unique classical solution $u \in H^\beta$ given by

$$u(t) = S_0(t)\zeta + \int_0^t S_0(t-\tau)\pi_0 g(\tau)d\tau + (K_1\pi_1 g)(t). \tag{A.10}$$

Moreover, if $g \in X^\beta$, the u given by (A.10) is in X^β. In either case $Y = X$ or H we have the estimate

$$||u||_{Y^\beta} \leq M_0(\beta)\|\zeta\|_Y + \tilde{M}(\beta)||g||_{Y^\beta} \tag{A.11}$$

where

$$\tilde{M}(\beta) := c_\pi \left(\frac{M_0(\beta/2)}{\beta/2} + M_1(\beta) \right).$$

Due to hypothesis (H1) and Lemma A.2, we have that in fact for any $\zeta \in H_0$ and $g \in H^\beta$, formula (A.10) defines a function $u \in H^\beta$ that satisfies the bound in (A.11). We will call this function u the *mild* solution of (A.9).

A.2 Nonlinear analysis with cutoff

In this section, we will consider the full nonlinear problem. The first observation is that by Lemma A.3, any classical solution u of (A.1) that is globally bounded in H must satisfy the equation

$$u(t) = S_0(t)\zeta + \int_0^t S_0(t-\tau)\pi_0 f(u(\tau))\, d\tau + (K_1\pi_1 f(u))(t), \tag{A.12}$$

where $\zeta = \pi_0 u(0) \in H_0$. The idea now is to use the contraction mapping theorem in the space H^β to prove the existence of a unique fixed point for the operator that yields the right hand side of Eq. (A.12). But the nonlinearity f may be only locally, not globally, Lipschitz. This forces us to localize Eq. (A.1) by changing the nonlinearity outside of a suitable neighborhood of 0 in the space H using a cutoff function. We will establish a center manifold theorem for the cutoff version of Eq. (A.1), and then show that this produces a local center manifold for the full problem.

Recall $f \in C^1(H, X)$ satisfies $f(0) = 0$, $Df(0) = 0$. Choose $\chi \in C^1(\mathbb{R}, [0, 1])$ such that

$$\chi(t) = \begin{cases} 1 & \text{for } t < 2, \\ 0 & \text{for } t > 3, \end{cases} \tag{A.13}$$

with $|\chi'(t)| \leq 2$ for all t. For $\delta > 0$, then define the cut-off nonlinearity by

$$f_\delta(x) := f(x) \cdot \chi\left(\frac{\|x\|_H}{\delta}\right) \tag{A.14}$$

for $x \in H$. Then f_δ is bounded, globally Lipschitz, and is C^1 near zero, with $f_\delta(x) = 0$ for $\|x\|_H > 3\delta$ and $Df_\delta(0) = 0$. We next define

$$L(f,\delta) := \sup\{\|Df(y)\|_{\mathcal{L}(H,X)} : y \in H, \|y\|_H < \delta\}, \tag{A.15}$$

$$L(f_\delta) := \sup\{\|f_\delta(x) - f_\delta(y)\|_X / \|x - y\|_H : x, y \in H \text{ with } x \neq y\}. \tag{A.16}$$

For any $\delta' > 0$ we have

$$\|f(u_1) - f(u_2)\|_X \leq L(f,\delta')\|u_1 - u_2\|_H \quad \text{for all } u_1, u_2 \in B(\delta'), \tag{A.17}$$

and since $Df \in C(H, \mathcal{L}(X))$ we have $L(f,\delta') \to 0$ as $\delta' \to 0^+$. Our cutoff yields the bounds

$$L(f_\delta) \leq 7L(f, 3\delta), \qquad L(f_\delta, \delta') \leq 3L(f,\delta') \quad \text{for } 0 < \delta' \leq \delta. \tag{A.18}$$

The cut-off version of Eq. (A.1) is

$$\frac{du}{dt}(t) = Au(t) + f_\delta(u(t)). \tag{A.19}$$

By Lemma A.3, any classical solution $u \in H^\beta$ of (A.19) must satisfy the equation

$$u(t) = T(\zeta, u)(t) := S_0(t)\zeta + \tilde{T}(u)(t) \tag{A.20}$$

where

$$\tilde{T}(u)(t) := \int_0^t S_0(t-\tau)\pi_0 f_\delta(u(\tau))d\tau + (K_1\pi_1 f_\delta(u))(t),$$

and where $\zeta = \pi_0 u(0) \in H_0$.

By Lemma A.3, if $\beta \in (0,\alpha)$ then $\tilde{T}: H^\beta \to X^\beta$ is Lipschitz continuous with Lipschitz constant $\tilde{M}(\beta)L(f_\delta)$, and in either case $Y = H$ or X we have that $T(\zeta,\cdot): Y^\beta \to Y^\beta$ is Lipschitz continuous with the same Lipschitz constant. Let δ_0 be chosen (depending on β) such that for $0 < \delta \leq \delta_0$,

$$\tilde{M}(\beta)L(f_\delta) < \tfrac{1}{2}. \tag{A.21}$$

Now we can solve Eq. (A.20) by the contraction mapping theorem in Y^β, and obtain

Proposition A.4 *(Fixed points of T) Let H, X, A, π_0, π_1, f and δ be as above. If $\beta \in (0,\alpha)$ is fixed, then the cut-off nonlinear problem (A.19) has for all $\zeta \in H_0$ a unique mild solution $u =: \tilde{u}(\zeta,\cdot) \in H^\beta$ of the form (A.20) with $\pi_0 u(0) = \zeta$. If $\zeta \in X_0$ then $\tilde{u}(\zeta,\cdot) \in X^\beta$ and is the unique classical solution in H^β. Moreover, in either case $Y = H$ or X, for any $\zeta_1, \zeta_2 \in Y_0$ we have*

$$||\tilde{u}(\zeta_1,\cdot) - \tilde{u}(\zeta_2,\cdot)||_{Y^\beta} \leq 2M_0(\beta)\|\zeta_1 - \zeta_2\|_Y. \tag{A.22}$$

Proof. We only need to show the estimates (the existence and uniqueness follow by the contraction mapping theorem). By estimates in Lemma A.3,

$$\begin{aligned} ||\tilde{u}(\zeta,\cdot) - \tilde{u}(\eta,\cdot)||_{Y^\beta} &\leq M_0(\beta)\|\zeta - \eta\|_Y + \tilde{M}(\beta)||f_\delta(\tilde{u}(\zeta,\cdot)) - f_\delta(\tilde{u}(\eta,\cdot))||_{Y^\beta} \\ &\leq M_0(\beta)\|\zeta - \eta\|_Y + \tilde{M}(\beta)L(f_\delta)||\tilde{u}(\zeta,\cdot) - \tilde{u}(\eta,\cdot)||_{Y^\beta}. \end{aligned}$$

□

Now we can obtain a global center manifold for the cut-off problem.

Proposition A.5 *(Global center manifold) Let H, X, A, π_0, π_1, β, δ and f be as in Proposition A.4. For $\zeta \in H_0$, let $\tilde{u}(\zeta, \cdot)$ denote the unique fixed point for $T(\zeta, \cdot)$ in H^β, and define*

$$\phi_\delta(\zeta) = \pi_1 \tilde{u}(\zeta, 0).$$

If $u \in H^\beta$ is a mild solution of Eq. (A.19), then $u(t) \in \widetilde{\mathcal{M}}_\delta$ for all $t \in \mathbb{R}$, where

$$\widetilde{\mathcal{M}}_\delta := \{\xi + \phi_\delta(\xi) : \xi \in H_0\}.$$

The map $\phi_\delta \in C_b(H_0, X_1) \cap \mathrm{Lip}(H_0, X_1)$, $\phi_\delta(0) = 0$, and $D\phi_\delta(0)$ exists and is zero. Also, for any $\delta' > 0$, for any $\zeta_1, \zeta_2 \in H$ with $\|\zeta_j\|_H < \delta'$ we have

$$\|\phi_\delta(\zeta_1) - \phi_\delta(\zeta_2)\|_X \leq L(\phi_\delta, \delta')\|\zeta_1 - \zeta_2\|_H$$

where $L(\phi_\delta, \delta') \to 0$ as $\delta' \to 0^+$ and with $C_* = 2M_0(\beta)M_1(\beta)c_\pi$ we have $L(\phi_\delta, \delta') \leq C_* L(f_\delta)$ for all $\delta' > 0$.

Proof. Let $\zeta \in H_0$, then the equation

$$u = S_0(\cdot)\zeta + \tilde{T}(u)$$

has the unique solution $u = \tilde{u}(\zeta, \cdot)$ in H^β. Since $\pi_0 \tilde{T}(u)(0) = 0$, we conclude that $\pi_0 u(0) = \zeta$. This means that if $u \in H^\beta$ is a mild solution of (A.19) then $u = \tilde{u}(\zeta, \cdot)$ and

$$\pi_1 u(0) = \pi_1 \tilde{u}(\zeta, 0) = \phi_\delta(\zeta).$$

In particular note the following. Fix any $s \in \mathbb{R}$. Then in hypothesis (H1), replacing g_1 by $g_1(s+\cdot)$ yields the solution $u_1(s+\cdot)$. Now, given $u = \tilde{u}(\zeta, \cdot)$ as above, define $v_s(t) = u(s+t)$. Since $\pi_1 u$ is a solution of (A.3) with $g_1(t) = \pi_1 f_\delta(u(t))$, we have that $\pi_1 v_s$ is a solution of (A.3) with $g_1(t) = \pi_1 f_\delta(v_s(t))$. Since $v_s \in H^\beta$, v_s is a mild solution of (A.19) with $\pi_0 v_s(0) = \pi_0 u(s)$. It follows that $v_s = \tilde{u}(\pi_0 u(s), \cdot)$ and therefore

$$u(s) = v_s(0) = \pi_0 u(s) + \phi_\delta(\pi_0 u(s)).$$

This proves that $u(t) \in \widetilde{\mathcal{M}}_\delta$ for all $t \in \mathbb{R}$.

To prove the estimate we let $\delta' > 0$ and suppose ζ_1, $\zeta_2 \in B(\delta')$. Note that we have $\phi_\delta(\zeta_j) = K_1 \pi_1 f_\delta(\tilde{u}(\zeta_j, \cdot))(0)$ and $\|\tilde{u}(\zeta_j, t)\|_H \leq 2M_0(\beta)e^{\beta|t|}\delta'$. Taking any $\gamma \in [\beta, \alpha)$, we compute

$$\begin{aligned}
\|\phi_\delta(\zeta_1) - \phi_\delta(\zeta_2)\|_X &\leq M_1(\gamma)c_\pi \|f_\delta(\tilde{u}(\zeta_1, \cdot)) - f_\delta(\tilde{u}(\zeta_2, \cdot))\|_{X^\gamma} \\
&\leq M_1(\gamma)c_\pi \sup_t \left(e^{-\gamma|t|} L(f_\delta, 2M_0(\beta)e^{\beta|t|}\delta') \|\tilde{u}(\zeta_1, t) - \tilde{u}(\zeta_2, t)\|_H \right) \\
&\leq 2M_0(\beta)M_1(\gamma)c_\pi \sup_t e^{-(\gamma-\beta)|t|} L(f_\delta, 2M_0(\beta)e^{\beta|t|}\delta') \|\zeta_1 - \zeta_2\|_H
\end{aligned}$$

Since $L(f_\delta, 2M_0(\beta)e^{\beta|t|}\delta')$ is uniformly bounded in t by $L(f_\delta)$ and it approaches zero as δ' does for t fixed, the asserted estimates follow, and $D\phi_\delta(0) = 0$. Clearly $\phi_\delta(0) = 0$. We also have that ϕ_δ is globally bounded, since

$$\|\phi_\delta(\zeta)\|_X \leq M_1(\gamma)c_\pi \sup_{u \in H} \|f_\delta(u)\|_X \leq 3\delta L(f_\delta) M_1(\gamma) c_\pi. \tag{A.23}$$

□

On the center manifold, the evolution reduces to a well-posed problem for a semilinear problem whose linear part is solved by the C^0-semigroup S_0.

Proposition A.6 *(Equation of evolution on the center manifold) Make the same hypotheses as in Proposition A.5. Let $\tilde{F}_\delta(w) := \pi_0 f_\delta(w + \phi_\delta(w))$ for $w \in H_0$. Then for any $\zeta \in H_0$, $u_0 = \pi_0\tilde{u}(\zeta, \cdot) \in H_0^\beta$ is the unique mild solution of*

$$\frac{d}{dt}u_0(t) = A_0 u_0(t) + \tilde{F}_\delta(u_0(t)) \quad (t \in \mathbb{R}), \qquad u_0(0) = \zeta. \tag{A.24}$$

If $\zeta \in X_0$, then $u_0 = \pi_0\tilde{u}(\zeta, \cdot) \in X_0^\beta$ is the unique classical solution for this problem. For any $T > 0$, the map $\zeta \mapsto u_0$ is Lipschitz continuous from H_0 to $C([-T, T], H)$.

Proof. Let $\zeta \in H_0$. Since $\tilde{F}_\delta$ is Lipschitz on H_0 and A_0 is the generator of a C^0 semigroup on H_0, existence and uniqueness of a global mild solution of (A.24) is proved in a standard way using semigroup theory [13]. By Proposition A.4, $\tilde{u}(\zeta, \cdot) \in H^\beta$ is the unique mild solution of (A.19) of the form (A.20) with $\pi_0\tilde{u}(\zeta, 0) = \zeta$. Now by Proposition A.5,

$$\tilde{u}(\zeta, t) = \pi_0\tilde{u}(\zeta, t) + \phi_\delta(\pi_0\tilde{u}(\zeta, t)).$$

Then by projecting (A.20) on the space H_0 we see that $\pi_0\tilde{u}(\zeta, \cdot) \in H_0^\beta$ is the unique mild solution of (A.24). If $\zeta \in X_0$ we have that $\pi_0\tilde{u}(\zeta, \cdot) \in X_0^\beta$ is the unique classical solution of (A.24). That the map $\zeta \mapsto u_0$ is Lipschitz continuous from H_0 to $C([-T, T], H)$ for any $T > 0$ follows in a standard way from the variation of parameters formulation of (A.24). $\square$

A.3 Proof of Theorem A.1.

Let δ be so small that both $\tilde{M}(\beta)L(f_\delta) < \frac{1}{2}$ and $C_* L(f_\delta) < \frac{1}{2}$. We take ϕ_δ as given by Proposition A.5. Then $\phi_\delta \in C_b(H_0, X_1) \cap \mathrm{Lip}(H_0, X_1)$. Let $\mathcal{M}_\delta \subset X$ be defined by

$$\mathcal{M}_\delta := \{\xi + \phi_\delta(\xi) : \xi \in X_0\}.$$

The statements in parts (i) and (ii) of the Theorem follow from Proposition A.5.

(iii) Let $y = \zeta + \phi_\delta(\zeta) \in \mathcal{M}_\delta$. Then there exists a unique classical solution $u = \tilde{u}(\zeta, \cdot) \in H^\beta$ of the cut-off Eq. (A.19) with $\pi_0\tilde{u}(\zeta, 0) = \zeta$. In consequence, $\phi_\delta(\zeta) = \pi_1\tilde{u}(\zeta, 0)$ and then $\tilde{u}(\zeta, 0) = \zeta + \phi_\delta(\zeta) = y$. Moreover, $u(t) \in \widetilde{\mathcal{M}}_\delta$ for all $t \in \mathbb{R}$. In other words, $u(t) = \pi_0 u(t) + \phi_\delta(\pi_0 u(t))$. Let J be any open interval containing 0 with $\pi_0(u(J)) \subset B(\delta)$. Then for any $t \in J$,

$$\|u(t)\|_H \le \|\pi_0 u(t)\|_H + \|\phi_\delta(\pi_0 u(t))\|_H \le (1 + L(\phi_\delta))\|\pi_0 u(t)\|_H \le 2\delta,$$

where $L(\phi_\delta)$ denotes the Lipschitz constant of ϕ_δ. Thus $f_\delta(u(t)) = f(u(t))$ for all $t \in J$. So u is a classical solution of the original Eq. (A.1) in J.

(iv) Let u_0 be a classical solution on $\mathbb{R}$ of

$$\frac{d}{dt}u_0(t) = A_0 u_0(t) + \pi_0 f(u_0(t) + \phi_\delta(u_0(t))),$$

such that $u_0(t) \in B(\delta) \cap X_0$ for all $t \in \mathbb{R}$. Let $\zeta = u_0(0) \in X_0$. Define w by $w = u_0 + \phi_\delta(u_0)$. We want to show that w is a classical solution of the full problem (A.1). Let $\tilde{u}(\zeta, \cdot) \in H^\beta$ be the unique classical solution of the cut-off Eq. (A.19). Then by Proposition A.5, $\tilde{u}(\zeta, \cdot)$ is the fixed point of $T(\zeta, \cdot)$ and $\pi_1\tilde{u}(\zeta, t) = \phi_\delta(\pi_0\tilde{u}(\zeta, t))$. Moreover, $\pi_0\tilde{u}(\zeta, \cdot)$ satisfies the above problem. By Proposition A.6 we conclude that $\pi_0\tilde{u}(\zeta, \cdot) = u_0$, and hence $\tilde{u}(\zeta, \cdot) = w(\cdot)$. On the other hand,

$$\|w(t)\|_H \le \|u_0(t)\|_H + \|\phi_\delta(u_0(t))\|_H \le (1 + L(\phi_\delta))\delta < 2\delta.$$

This implies that $f_\delta(w(t)) = f(w(t))$ for all $t \in \mathbb{R}$ and w is a classical solution of the full problem (A.1) on $\mathbb{R}$.

(v) Let u be a classical solution for the full problem (A.1) such that $\|u(t)\|_H \le \delta$ for all $t \in \mathbb{R}$. Then u is also a solution of the cut-off Eq. (A.19) in H^β. Define u_0 as $u_0(t) = \pi_0 u(t)$. Then by Proposition A.5 we conclude that $u(t) = u_0(t) + \phi_\delta(u_0(t))$. Since $u(t) \in X$ we conclude that $u(t) \in \mathcal{M}_\delta$ for all t. $\square$

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References

[1] P. Bates and C. K. R. T. Jones. Invariants manifolds for semilinear partial differential equations. *Dynamics Reported*, 2:1–38, 1989.

[2] D. J. Benney and J. C. Luke. Interactions of permanent waves of finite amplitude. *J. Math. Phys.*, 43:309–313, 1964.

[3] W. Craig and D. P. Nicholls. Traveling two and three dimensional capillary gravity water waves. *SIAM J. Math. Anal.*, 32(2):323–359, 2000.

[4] M. D. Groves. An existence theory for three-dimensional periodic travelling gravity-capillary water waves with bounded transverse profiles. *Physica D*, 152:395–415, 2001.

[5] M. D. Groves and A. Mielke. A spatial dynamics approach to three-dimensional gravity-capillary steady water waves. *Proc. Royal Soc. Edin. A*, 131(1):83–136, 2001.

[6] J. Hammack, D. McCallister, N. Scheffner, and H. Segur. Two dimensional periodic waves in shallow water II: Asymmetric waves. *J. Fluid Mech.*, 285:95–122, 1995.

[7] J. Hammack, N. Scheffner, and H. Segur. Two dimensional periodic waves in shallow water. *J. Fluid Mech.*, 209:567–589, 1989.

[8] M. Haragus-Courcelle and R.L. Pego. Spatial wave dynamics of steady oblique wave interactions. *Physica D*, 145(3–4):207–232, 2000.

[9] A. Kirchgässner. Wave solutions of reversible systems and applications. *J. Diff. Eqns*, 45:113–127, 1982.

[10] A. Mielke. Reduction of quasilinear elliptic equations in cylindrical domains with applications. *Math. Meth. Appli. Sci.*, 10:51–66, 1988.

[11] A. Mielke. On nonlinear problems of mixed type: a qualitative theory using infinite-dimensional center manifolds. *J. Dyn. Diff. Eqns.*, 4(3):419–443, 1992.

[12] P. A. Milewski and J. B. Keller. Three dimensional water waves. *Studies Appl. Math.*, 37:149–166, 1996.

[13] A. Pazy. *Semigroups of linear operators and applications. Appl. Math. Sci.*, volume 44. Springer–Verlag, NY, 1983.

[14] R. L. Pego and J. R. Quintero. Two-dimensional solitary waves for a Benney-Luke equation. *Physica D*, 132:476–496, 1999.

[15] B. Scarpellini. On nonlinear problems of mixed type: a qualitative theory using infinite-dimensional center manifold. I and II. *J. Appl. Math. Phys. (ZAMP)*, 42:1–32, 280–314, 1990.

[16] A. Vanderbauwhede and G. Iooss. Center manifold theory in infinite dimensions. *Dynamics Reported, New Series*, 1:125–163, 1992.